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CERTAIN VARIATIONAL PROBLEMS WITH A SMALL PARAMETER IN THE THEORY OF ELASTICITY *

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For the mixed problem of elasticity theory on the deformation of a transversely isotropic cylinder, it is proved than in the selection of a specific small parameter the zeroth approximation equations agree with the bending equations for an elastic isotropic plate based on the Kirchhoff-Love hypotheses /1/. Certain singularly perturbed contact problems of the Signorini type are considered.

1. We consider an elastic transversely-isotropic cylinder $Q = \omega\cdot(-h/2, h/2)$ where ω is a bounded domain in the (x_1, x_2) plane with fairly smooth boundary γ. We select the small parameter ε as follows: $\varepsilon = \sqrt{E/E'}$, where E is Young's modulus of the material in the plane of isotropy of the material $x_3 \equiv \text{const}$, and E' is Young's modulus in the orthogonal direction. In a real physical situation the parameter ε is small for an elastic cylinder reinforced by a family of boron or carbon fibres in the direction of the vertical axis, whose axial Young's modulus is considerably greater than Young's modulus in the circumferential direction.

We divide the stress by Young's modulus E, we retain the same notation for these dimensionless stresses and using Hooke's law we express the stress in terms of the strain

$$\begin{aligned}
&\sigma_{11} = a_{11}e_{11} + a_{12}e_{22} + a_{13}e_{33}, \quad \sigma_{12} = 2(1+\nu)^{-1}e_{12} \\
&\sigma_{22} = a_{12}e_{11} + a_{11}e_{22} + a_{13}e_{33}, \quad \sigma_{13} = 2b\varepsilon^{-2}e_{13} \\
&\sigma_{33} = a_{13}(e_{11} + e_{22}) + a_{33}\varepsilon^{-2}e_{33}, \quad \sigma_{23} = 2b\varepsilon^{-2}e_{23}
\end{aligned} \tag{1.1}$$

Here

$$\begin{aligned}
&a_{11} = (1-\mu^2\varepsilon^2)(1+\nu)^{-1}a_0^{-1}, \quad a_{12} = (\nu + \mu^2\varepsilon^2)(1+\nu)^{-1}a_0^{-1} \\
&a_{13} = \mu a_0^{-1}, \quad a_{33} = (1-\nu)a_0^{-1}, \quad a_0 = 1 - \nu - 2\mu^2\varepsilon^2 \\
&e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad i, j = 1, 2, 3
\end{aligned}$$

(summation is from 1 to 3 over repeated subscripts), $b = E'/G'$ is the ratio between Young's modulus E' and the shear modulus G' in a direction orthogonal to the plane of isotropy, ν is Poisson's ratio in the plane of isotropy, and μ is the secondary Poisson's ratio. The positivity of the strain potential energy results in the constraints $0 < \nu < 1$, $a_0 > 0$, $b > 0$.

We examine the mixed boundary value problem for the system of equations of elasticity theory

$$\begin{aligned}
&-\frac{\partial \sigma_{ij}}{\partial x_j} + f_i = 0, \quad f_i \in L_2(G), \quad i = 1, 2, 3 \\
&\sigma_{i3}|_{x_3=\pm h/2} = 0, \quad u_i|_{\gamma\cdot(-h/2, h/2)} = 0
\end{aligned} \tag{1.2}$$

Later, in order to note clearly the dependence of the solution of problem (1.2) on the parameter ε, we shall mark the stress and displacement by the superscript ε, here denoting the stress and displacement in the limit problem by σ_{ij}° and u_i°. We introduce a Hilbert space with the standard norm

$$\begin{aligned}
&V = \{u;\, u = (u_1, u_2, u_3),\, u_k \in H^1(Q),\, u_k|_{\gamma\times(-h/2, h/2)} = 0, \quad k = 1, 2, 3\} \\
&\|u\|_V = \left\{\int_Q (u_{i,k}u_{i,k} + u^2)\, dx\right\}^{1/2}
\end{aligned}$$

The generalized solution of problem (1.2) is determined in a standard manner /2/ as a function $u^\varepsilon \in V$ such that for any function $v \in V$ the following integral identity is satisfied:

$$\begin{aligned}
&a^\varepsilon(u^\varepsilon, v) = L(v); \quad L(v) = \int_Q f_k v_k\, dx_1\, dx_2\, dx_3 \\
&a^\varepsilon(u^\varepsilon, v) = \frac{1}{2}\int_Q \sigma_{ij}(u^\varepsilon)\, e_{ij}(v)\, dx_1\, dx_2\, dx_3
\end{aligned} \tag{1.3}$$

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We will study the question of the behaviour of the solution of the variational problem (1.3) in the limit as $\varepsilon \to +0$.

Lemma 1. For sufficiently small $\varepsilon > 0$ a constant $c > 0$ exists, independent of ε, such that

$$a^\varepsilon(u^\varepsilon, u^\varepsilon) \geqslant c \| u^\varepsilon \|_V^2 \tag{1.4}$$

For the proof we estimate the least eigenvalue of the matrix of the elastic constants for e_{11}, e_{22}, e_{33} in (1.1) independent of ε . It is known that all the eigenvalues are positive for a symmetric positive-definite matrix; the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of this matrix have the following orders:

$$\lambda_1 = (1+\nu)^{-1}, \quad \lambda_2 \sim a_0^{-1}, \quad \lambda_3 \sim (1-\nu)\varepsilon^{-2} a_0^{-1}$$

Obviously, $\lambda_2, \lambda_3 > \lambda_1$; let $\varepsilon < \varepsilon_0 = 2^{-1/2}(1-\nu)^{1/2}\mu^{-1}$. Then

$$a^\varepsilon(u^\varepsilon, u^\varepsilon) \geqslant c_0 \int_Q e_{ij}(u^\varepsilon) e_{ij}(u^\varepsilon)\, dx_1 dx_2 dx_3 \geqslant c \| u^\varepsilon \|_V^2, \tag{1.5}$$

$$c_0 \doteq \min\left(\lambda_{1/4}, \frac{1+\nu}{2}, b\varepsilon_0^{-2}\right)$$

The validity of the second inequality in (1.5) follows from the validity of the second Korn inequality in V /2/.

Lemma 2. For sufficiently small $\varepsilon > 0$ a unique solution of problem (1.3) exists, where the following estimates, uniform in ε, hold

$$\| u^\varepsilon \|_V \leqslant c_1, \quad \| \sigma_{ij}(u^\varepsilon) \|_{L_2} \leqslant c_2, \quad i, j = 1, 2 \tag{1.6}$$

$$\varepsilon^{-2} \| e_{i3}(u^\varepsilon) \|_{L_2}^2 \leqslant c_3, \quad i = 1, 2, 3$$

Indeed, the right side of (1.3) is a continuous linear functional in V, where by virtue of the embedding theorem $| L(u^\varepsilon) | \leqslant c_4 \| u^\varepsilon \|_V$. But $a^\varepsilon(u^\varepsilon, u^\varepsilon) \geqslant c \| u^\varepsilon \|_V^2$ from (1.5), and consequently, $\| u^\varepsilon \|_V \leqslant c_1$. The quadratic form $a^\varepsilon(u^\varepsilon, u^\varepsilon)$ is, by virtue of positive-definiteness, represented as the sum of squares of linear forms of the strain; since the right side of (1.3) is estimated by a constant independent of ε, we obtain the remaining estimates in (1.6).

It follows from Lemma 2 that a subsequence (for which we keep the previous notation) can be extracted from the sequence u^ε such that u^ε will converge weakly to a certain element $u^\circ \in V$, $\sigma_{ij}(u^\varepsilon)$ converges weakly to $\sigma_{ij}(u^\circ)$ in V $(i, j = 1, 2)$, and $e_{i3}(u^\varepsilon) \to 0$ strongly in $L_2(Q)$, $i = 1, 2, 3$.

Lemma 3. The functions $u_1^\circ, u_2^\circ, u_3^\circ$ are representable in the form

$$u_3^\circ = u_3^\circ(x_1, x_2), \; u_3^\circ \in H_0^2(\omega) \tag{1.7}$$

$$u_k^\circ(x_1, x_2, x_3) = g_k(x_1, x_2) - x_3 \frac{\partial u_3^\circ}{\partial x_k}(x_1, x_2) \tag{1.8}$$

$$g_k(x_1, x_2) \in H_0^1(\omega), \quad k = 1, 2$$

It actually follows from $\| u^\varepsilon_{3, x_3} \| \leqslant c\varepsilon$ that $\partial u_3^\circ / \partial x_3 = 0$, and consequently u_3° is independent of x_3. Furthermore, $\| u^\varepsilon_{3, x_k} + u^\varepsilon_{k, x_3} \|_{L_2} \leqslant c\varepsilon$, $k = 1, 2$, therefore, $\overset{\circ}{u}_{k, x_3} = -\overset{\circ}{u}_{3, x_k}$ and if $g_k(x_1, x_2)$ denotes the trace of the function u_k° in the plane $\{(x_1, x_2) \in \omega, x_3 = 0\}$, we obtain (1.8). Functions belonging to $H^1(Q)$ and equal to zero on the side surface are on the left side of the second equation, hence $u_3^\circ \in H_0^2(\omega)$ and $g_k \in H_0^1(\omega)$, $k = 1, 2$.

Let V_{KL} denote a closed subspace of V separated out by the conditions $e_{k3}(v) = 0$, $k = 1, 2, 3$. It is known /3/ that V_{KL} is isomorphic to $[H_0^1(\omega)]^2 \times H_0^2(\omega)$.

Let us examine the integral identity (1.3) in the functions $v \in V_{KL}$. Passing to the limit in the subsequence already selected and integrating with respect to x_3, we obtain the integral identity

$$\frac{h}{1-\nu^2} \int_\omega [(\varepsilon_{11}(g) + \nu\varepsilon_{22}(g))\varepsilon_{11}(\psi) + (\nu\varepsilon_{11}(g) + \varepsilon_{22}(g))\varepsilon_{22}(\psi) + 4(1-\nu)\varepsilon_{12}(g)\varepsilon_{12}(\psi)]\, dx_1 dx_2 + \tag{1.9}$$

$$\frac{h^3}{12(1-\nu^2)} \int_\omega [(\delta_{11}(u_3^\circ) + \nu\delta_{22}(u_3^\circ))\delta_{11}(\psi_3) + (\nu\delta_{11}(u_3^\circ) + \delta_{22}(u_3^\circ))\delta_{22}(\psi_3) + 4(1-\nu)\delta_{12}(u_3^\circ)\delta_{12}(\psi_3)]\, dx_1 dx_2 =$$

$$h \int_\omega \sum_{k=1}^{2} \langle f_k \rangle \psi_k\, dx_1 dx_2 + \int_\omega \sum_{k=1}^{2} \langle g_k \rangle \frac{\partial \psi_3}{\partial x_k}\, dx_1 dx_2$$

$$\psi = (\psi_1, \psi_2, \psi_3) \in [H_0^1(\omega)]^2 \times H_0^2(\omega)$$

$$\varepsilon_{ij}(g) = \frac{1}{2}(g_{i,j} + g_{j,i}), \; \varepsilon_{ij}(\psi) = \frac{1}{2}(\psi_{i,j} + \psi_{j,i}), \quad i, j = 1, 2$$

$$\delta_{11}(\psi)=\frac{\partial^2\psi}{\partial x_1^2},\quad \delta_{22}(\psi)=\frac{\partial^2\psi}{\partial x_2^2},\quad \delta_{12}(\psi)=2\frac{\partial^2\psi}{\partial x_1\partial x_2}$$
$$g_k=x_3f_k,\quad k=1,2$$

($\langle f_k\rangle$, $\langle g_k\rangle$ are averages over the plate thickness). The existence and uniqueness of the solution of the limit problem are well known /3,4/. It hence follows that the whole initial sequence u^ε converges weakly to u°, i.e., the following holds:

Theorem 1. The solution of problem (1.3) converges weakly to the solution of the limit problem (1.9).

Apart from the coefficients, the variational problem (1.9) corresponds to the problem of the compression-tension and bending of a thin isotropic slab. However, it must be noted that the smallness of the normal stresses is not proved successfully here.

Theorem 2. As $\varepsilon\to+0$ the sequence u^ε converges strongly to u° in V.

Indeed, we have

$$a^\varepsilon(u^\varepsilon-u^\circ,u^\varepsilon-u^\circ)=L(u^\varepsilon-u^\circ)-a_1^\varepsilon(u^\circ,u^\varepsilon-u^\circ)\tag{1.10}$$
$$L(u^\varepsilon-u^\circ)=\int_Q f_k(u_k^\varepsilon-u_k^\circ)\,dx_1\,dx_2\,dx_3,$$
$$a_1^\varepsilon(u^\circ,u^\varepsilon-u^\circ)=\frac{1}{2}\int_Q\Big[(a_{11}e_{11}(u^\circ)+a_{12}e_{22}(u^\circ)e_{11}(u^\varepsilon-u^\circ)+$$
$$(a_{12}e_{11}(u^\circ)+a_{11}e_{22}(u^\circ)e_{22}(u^\varepsilon-u^\circ)+\frac{4}{1+\nu}e_{12}(u^\varepsilon-u^\circ)e_{12}(u^\circ)]\,dx_1\,dx_2\,dx_3$$

Since u^ε converges weakly to u° in V, then by the Rellich embedding theorem u^ε converges strongly to u° in $[L_2(Q)]^3$, and consequently $L(u^\varepsilon-u^\circ)\to0$ as $\varepsilon\to+0$; from the weak convergence of $e_{ij}(u^\varepsilon)$ to $e_{ij}(u^\circ)(i,j=1,2)$ in $[L_2(Q)]^3$ it follows that $a_1^\varepsilon(u^\circ,u^\varepsilon-u^\circ)\to0$ as $\varepsilon\to+0$. On the other hand

$$a^\varepsilon(u^\varepsilon-u^\circ,u^\varepsilon-u^\circ)\geqslant c\,\|u^\varepsilon-u^\circ\|_V^2$$

and from the convergence of the right side of (1.10) to zero there follows the convergence of the left to zero; consequently $\|u^\varepsilon-u^\circ\|_V\to0$ as $\varepsilon\to+0$.

2. Certain singularly perturbed inequalities can also be studied. As an example, we consider the following Signorini problem. Let K be a closed convex cone in $W=\{u=(u_1,u_2,u_3),\ u_k=0$ on $\gamma\cdot(-h/2,h/2),\ u_k\in H^1(Q),\ k=1,2,3\}$, defined by the condition $u_3\leqslant0$ on the lower base Γ_0 of a cylinder. We examine the asymptotic behaviour of the solution of the inequality

$$a^\varepsilon(u^\varepsilon,v-u^\varepsilon)\geqslant(f,v-u^\varepsilon)\quad\forall v\in K,\quad f_1=f_2=0\tag{2.1}$$

as $\varepsilon\to+0$.

We associate a problem with a penalty with the inequality (2.1)

$$a^\varepsilon(u^{\varepsilon,\eta},v)+\frac{1}{\eta}\int_{\Gamma_0}[u_3^{\varepsilon,\eta}]^+v_3\,dx_1\,dx_2=(f,v),\quad\eta>0\tag{2.2}$$

We will study the behaviour of the non-linear problem (2.2) as $\varepsilon\to+0$. In this case the estimates (1.6) are conserved; consequently, a subsequence $u^{\varepsilon_n,\eta}$ can be extracted from the sequence $u^{\varepsilon,\eta}$ such that $u^{\varepsilon_n,\eta}\to u^{0,\eta}$ weakly in W and strongly in $[L_2(Q)]^3$, $e_{i3}(u^{\varepsilon,\eta})\to0$ $(i=1,2,3)$ strongly in $L_2(Q)$ and the limit functions $u^{0,\eta}$ have the form (1.7) and (1.8). The family of traces of the function $u^{\varepsilon,\eta}$ is here strongly compact in $L_2(\Gamma)$, where Γ is the boundary of Q. Since $f_1=f_2=0$, $g_1(x)=g_2(x)=0$, the sole function different from zero is $u_3^{0,\eta}\in H_0^2(\omega)$.

We examine the integral identity (2.2) in the subspace V_{KL}, we pass to the limit as $\varepsilon\to+0$, and we integrate over the height. We consequently obtain that $u_3^{0,\eta}$ satisfies the integral identity

$$b(u_3^{\circ,\eta},v_3)+\frac{1}{\eta}\int_\omega[u_3^{\circ,\eta}]^+v_3\,dx_1\,dx_2=h\int_\omega\langle f_3\rangle\,dx_1\,dx_2\tag{2.3}$$

where $b(u,\psi)$ is a bilinear form that is second on the left side of (1.9). Problem (2.3) is exactly the problem with a penalty on the contact of a plate with a stiff undeformable base.

The existence of a unique solution of problem (2.3) is well known. Hence, the whole sequence $u^{\varepsilon,\eta}$ converges to $u^{0,\eta}$. Passing to the limit as $\eta\to+0$ in (2.3), we obtain that $u_3^{0,0}$ satisfies the variational inequality

$$b(u_3^{\circ,\circ},v_3-u_3^{\circ,\circ})\geqslant(g_3,v_3-u_3^{\circ,\circ}),\quad\forall v_3\in K\cap V_{KL}\tag{2.4}$$

Passage to the limit as $\eta\to+0$ is given a foundation by known methods.

The following therefore holds:

Theorem 3. As $\varepsilon \to +0$ the solution of the variational inequality (2.1) converges weakly to the solution of the variational inequality (2.4).

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THE SUFFICIENT CONDITIONS FOR AN EXTREMUM IN PROBLEMS OF OPTIMIZING THE SHAPES OF ELASTIC PLATES *

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The problems of selecting the thickness distributions of elastic plates in order to maximize the fundamental free vibrations frequency, as well as to minimize the strain potential energy, are considered necessary and sufficient conditions are obtained for a weak local extremum in the such optimal design problems. These conditions retain their form even for reciprocal problems: minimization of plate weight when there are constraints on the fundamental frequency or the strain potential energy. The conditions obtained include an integral estimate on the maximum growth of second derivatives of the thickness distributions that satisfy the necessary extremum conditions.

Problems on optimizing the shape of elastic plates have been solved numerically /1-8/. It has been proved /9/ that these problems cannot have a strong extemum. It is shown /10,11/ that for solutions to exists it is sufficient to improve integral constraints on the nature of the growth of the derivatives of the allowable thickness distributions.

1. Formulation of the problem.

Consider a plate of variable thickness $h(x, y)$ clamped along a piecewise-smooth contour Γ bounding the domain D in the xy plane. Let S be the area of the domain D and V the volume of the plate. In the undeformed state the plate middle surface coincides with the domain D. The plate is simply supported on the part Γ_1 of the boundary Γ, and rigidly clamped on the remaining part Γ_2. The function of plate deflections is denoted by $w(x, y)$. We introduce the dimensionless variables

$$x' = xS^{-1/2}, \quad y' = yS^{-1/2}, \quad h'(x, y) = h(x, y)\, SV^{-1} \tag{1.1}$$

The problem of the frequencies of free vibrations has the following form in the notation used (we omit the primes on the dimensionless variables):

$$A(h)\, w(x, y) = \lambda h w(x, y), \quad \lambda = 12(1-\nu^2) E^{-1} S^4 V^{-2} \omega^2 \tag{1.2}$$

$$(w)_\Gamma = 0\left(\frac{\partial w}{\partial n}\right)_{\Gamma_2} = 0, \quad \left(h^3\left(\Delta w - \frac{1-\nu}{R}\frac{\partial w}{\partial n}\right)\right)_{\Gamma_1} = 0 \tag{1.3}$$

$$A(h) = \frac{\partial^2}{\partial x^2} h^3\left(\frac{\partial^2}{\partial x^2} + \nu\frac{\partial^2}{\partial y^2}\right) + \frac{\partial^2}{\partial y^2} h^3\left(\frac{\partial^2}{\partial y^2} + \nu\frac{\partial^2}{\partial x^2}\right) + 2(1-\nu)\frac{\partial^2}{\partial x\, \partial y} h^3 \frac{\partial^2}{\partial x\, \partial y} \tag{1.4}$$

Here E is Young's modulus, ν is Poisson's ratio, ω is the frequency of free vibrations, $\partial w/\partial n$ is the derivative with respect to the external normal to Γ, R is the radius of curvature, and Δ is the Laplace operator.

In the variables (1.1) the static bending problem of a plate loaded by a transverse force $p(x, y)$ has the form

$$A(h)\, w(x, y) = q(x, y), \quad q = 12(1-\nu^2) E^{-1} S^{-5/2} V^{-3} p(x, y) \tag{1.5}$$

where the differential operator $A(h)$ is given by (1.4), and the function w satisfies the

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